GCSE Maths – Geometry and Measures

Vector Operations

Notes

WORKSHEET

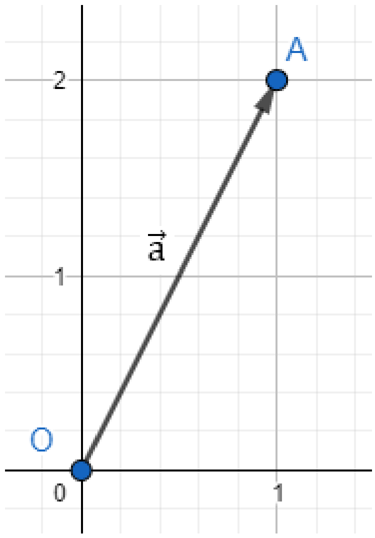
Vector Operations

Representation of Vectors

A **vector** is defined as having both a **magnitude** (size) and **direction**. It can be used to connect two different points in space.

The **vector** connecting the **origin**, denoted O, to point A is denoted $\overrightarrow{OA}$, $\vec{a}$ or $\boldsymbol{a}$. However, when writing, to indicate the **bold vector**, we underline the vector instead as such: $\underline{a}$. This **vector** tells us how to get from the **origin** to point A.

Diagrammatically, **vectors** are represented using a line with an arrow connecting two points. Below is an example of vector $\vec{a}$ when point A is $(1,2)$.



Column vector notation

Instead of traditional (x, y) notation that we use for describing points, we use **column vector** notation for describing vectors: $\binom{x}{y}$.

- The **x-value** tells us how many units to move in the x direction.
- The **y-value** tells us how many units to move in the y direction.

If either value x or y is **negative**, we move in the **negative** x or y direction, respectively.

The **vector** $\vec{a}$ above is represented in **column** notation as $\binom{1}{2}$.
This tells us to move 1 unit in the positive x direction and 2 units in the positive y direction.

Since **vectors** do not specify a starting point, the **vector** that takes us from the point $(2,1)$ to $(3,4)$ is the **same vector** as the one that takes us from $(0,0)$ to $(1,3)$ as both **vectors** are represented as $\binom{1}{3}$. Diagrammatically, this is like **shifting the starting position** of the **vector** from $(2,1)$ to $(0,0)$.

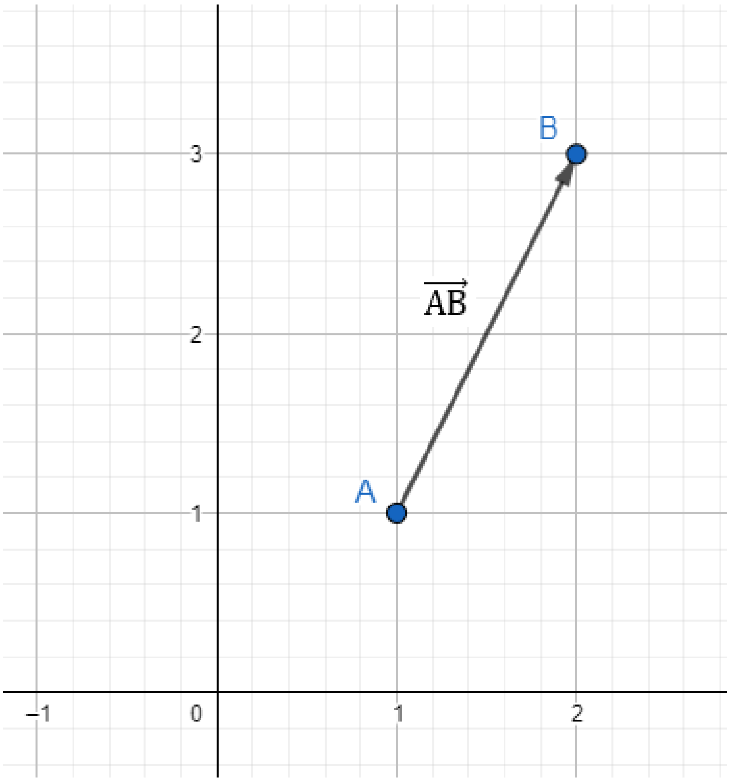
Connecting Two Points with a Vector

If we have a point A and a point B, we can connect the two points using a **vector** denoted $\overrightarrow{\boldsymbol{AB}}$ which tells us how to get from point A to point B.

If point A is $(1,1)$ and point B is $(2,3)$ then we need to move 1 unit right in the **x-coordinate** direction and 2 units up in the **y-coordinate** direction to get from A to B.
So,

$$\overrightarrow{\boldsymbol{AB}} = \binom{1}{2}.$$

This is represented **diagrammatically** below.



Adding and Subtracting Vectors Diagrammatically

Vectors can be **added** and **subtracted** diagrammatically. To **add** a **vector**, we follow it along the vector arrow from start to finish. To **subtract** a **vector**, we go **backwards** on the vector arrow.

So, if we were at point A and applied the **vector** $\overrightarrow{AB}$ then we would end up at point B. But if we were at point B and applied the **negative vector** $-\overrightarrow{AB}$ then we would go **backwards** on the **vector** arrow $\overrightarrow{AB}$ and end up at point A.

Using this information, we can prove that the **vector** $\overrightarrow{AB}$ can also be expressed as

$$\overrightarrow{OB} - \overrightarrow{OA}.$$

To see why, let's look at the diagram below.

To get from A to B, we could go straight from A to B resulting in the **vector** $\overrightarrow{AB}$. However, we could also go from A to O and then O to B resulting in $\overrightarrow{AO} + \overrightarrow{OB}$.

To get from A to O is the exact **opposite** of getting from O to A. If $\overrightarrow{OA} = \binom{1}{1}$, then $\overrightarrow{AO} = -\binom{1}{1} = \binom{-1}{-1}$ which is the same as **negative** $\overrightarrow{OA}$. Therefore, we can write

$$\overrightarrow{AO} = -\overrightarrow{OA}.$$

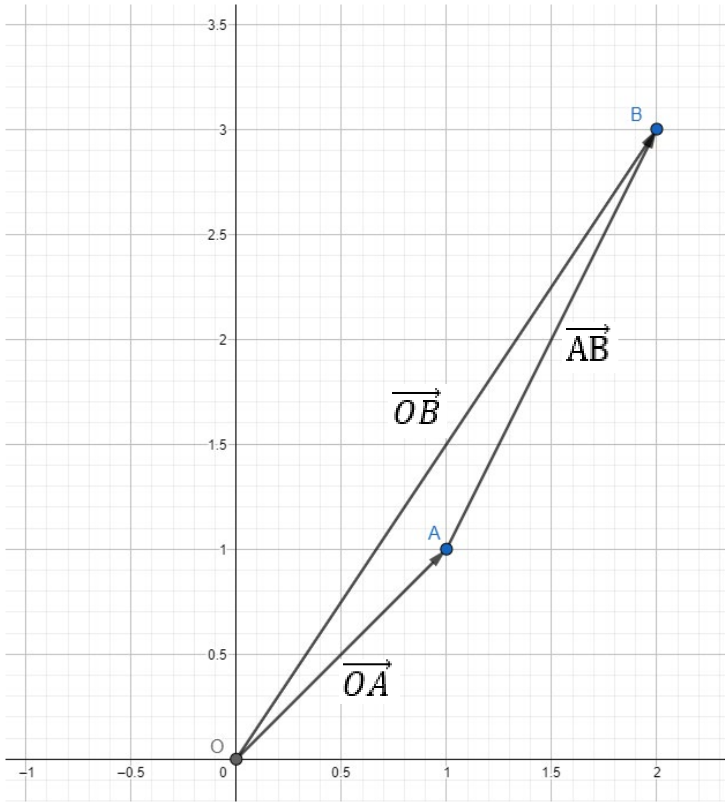
So,

$$\overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB}.$$

This means that,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}.$$

Rewriting vectors like this can be very useful for solving problems.



Adding and Subtracting Vectors in Column Notation

To **add** and **subtract** vectors in **column vector** notation, we add each of the coordinate rows.

Suppose we had **vector** $\boldsymbol{a} = \binom{a_x}{a_y}$ and **vector** $\boldsymbol{b} = \binom{b_x}{b_y}$ then $\boldsymbol{a} + \boldsymbol{b} = \binom{a_x + b_x}{a_y + b_y}$.

Similarly, for **subtraction**, $\boldsymbol{a} - \boldsymbol{b} = \binom{a_x - b_x}{a_y - b_y}$.

Example: Find $\boldsymbol{a} + \boldsymbol{b}$ if $\boldsymbol{a} = \binom{5}{-2}$, $\boldsymbol{b} = \binom{-3}{0}$

1. Write $\boldsymbol{a} + \boldsymbol{b}$ as one **column vector** by **adding** each of the rows.

$$\boldsymbol{a} + \boldsymbol{b} = \binom{5}{-2} + \binom{-3}{0} = \binom{5 + -3}{-2 + 0}$$

2. **Sum** each row and calculate the total.

$$5 + -3 = 2$$
$$-2 + 0 = -2$$

$$\boldsymbol{a} + \boldsymbol{b} = \binom{5 + -3}{-2 + 0} = \binom{2}{-2}$$

Multiplying Vectors by a Scalar

A **scalar** is a numerical value that has a **magnitude** (a size) but **no direction**. For example, the values $2, 5, -3, 0, \frac{3}{4}$ are all **scalars**.

To multiply a **vector** by a **scalar**, we separately multiply each row of the **vector** by the **scalar**.

Suppose we had **vector** $\boldsymbol{a} = \binom{a_x}{a_y}$ and we multiplied it by the **scalar** k:

$$k\boldsymbol{a} = k\binom{a_x}{a_y} = \binom{ka_x}{ka_y}$$

Example: What is vector $\boldsymbol{b}$ if $\boldsymbol{b} = 3\boldsymbol{a}$ and $\boldsymbol{a} = \binom{-2}{5}$?

1. **Multiply** each **row** of **vector** $\boldsymbol{a}$ by the **scalar** 3.

$$3\boldsymbol{a} = 3\binom{-2}{5} = \binom{3(-2)}{3(5)}$$

2. **Calculate** each new value and write the new **vector** $\boldsymbol{b}$.

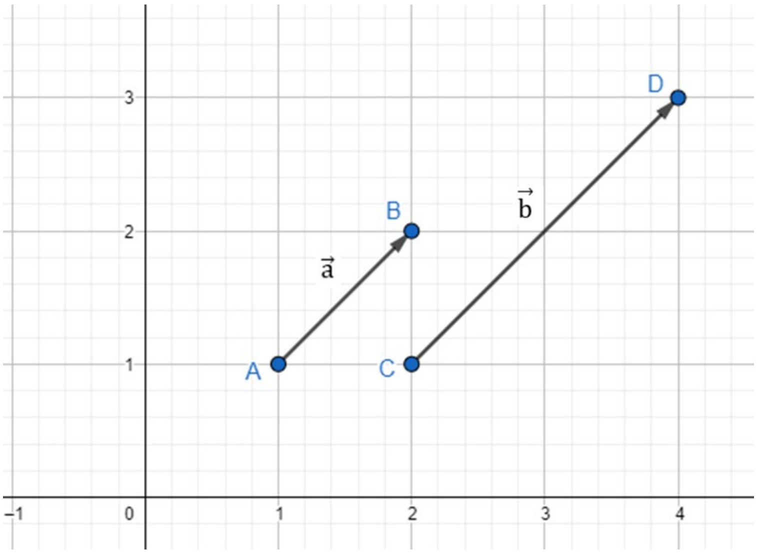
$$3 \times -2 = -6$$
$$3 \times 5 = 15$$

$$\boldsymbol{b} = \binom{3(-2)}{3(5)} = \binom{-6}{15}$$

Diagrammatic Effect of Multiplying by a Scalar

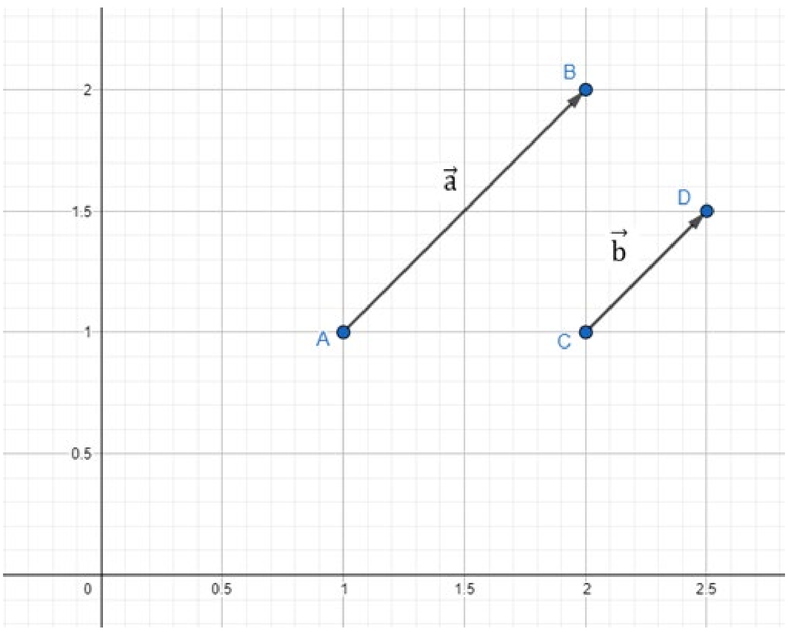
Multiplying a **vector** by **scalar** has different effects on the **diagrammatic** representation of the **vector,** depending on the **scalar** it is **multiplied** by.

If the vector is **multiplied** by a **positive scalar** k that is more than 1 $(k > 1)$, then the vector is **elongated** (stretched) by that factor. Below is an example of a vector $\vec{a} = \binom{1}{1}$ being **multiplied** by 2 to become vector $\vec{b} = \binom{2}{2}$.



However, if the vector is **multiplied** by a **positive scalar** k that is **less** than 1 $(0 < k < 1)$, then the vector becomes **squashed** and shrinks by that factor.

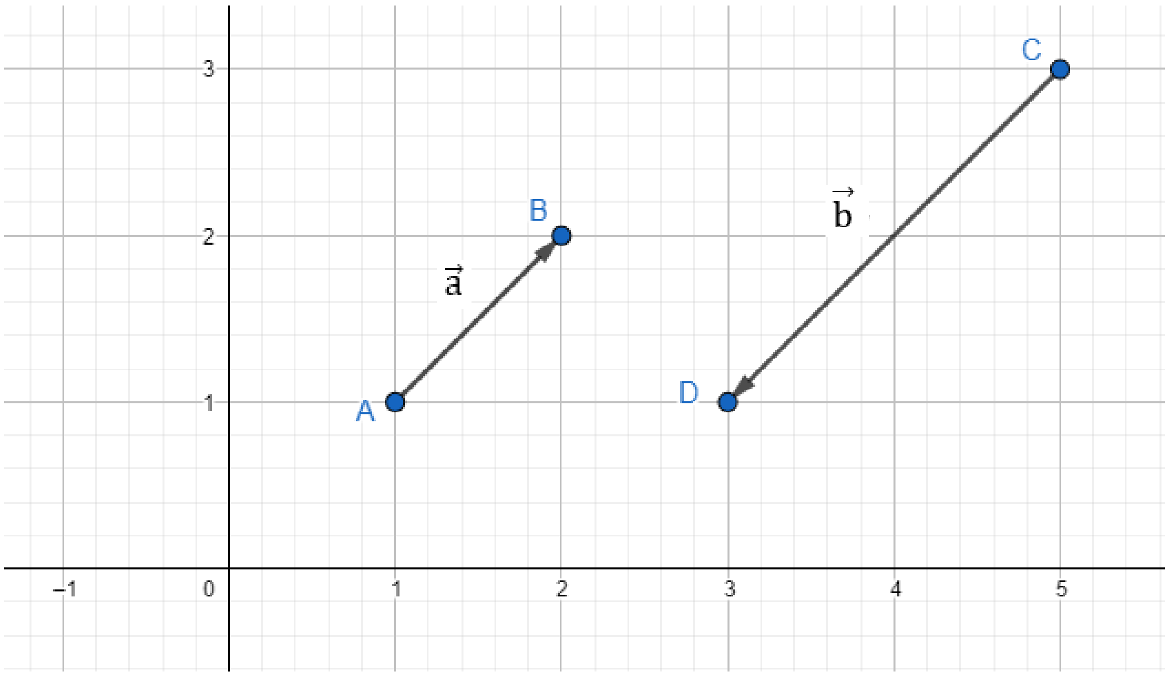
Below is an example of a **vector** $\vec{a} = \binom{1}{1}$ being **multiplied** by $\frac{1}{2}$ to become **vector** $\vec{b} = \binom{0.5}{0.5}$.



Lastly, if a **vector** is **multiplied** by **negative** scalar k, then the **vector** changes to the **opposite** direction.

Similar to the positive case, if the negative scalar k is less than -1 $(k < -1)$ then the vector elongates. If the negative scalar k is greater than -1 $(-1 < k < 0)$ then the vector shrinks.

Below is an example of a **vector** $\vec{a} = \binom{1}{1}$ being **multiplied** by -2 to become **vector** $\vec{b} = \binom{-2}{-2}$. Notice how the **arrow** is pointing in the **opposite** direction.



Vector Operations – Practice Questions

1. Given the vectors $\boldsymbol{a} = \binom{1}{4}$ $\boldsymbol{b} = \binom{-3}{6}$ $\boldsymbol{c} = \binom{-1}{-2}$

   Write the following expressions as a single column vector.

   a) **a** + **b**

   b) 3**a** – 2**c**

   c) 4**a** – **b** + 2**a**

2. Let $\boldsymbol{a} = \binom{1}{6}$ and $\boldsymbol{b} = \binom{3}{-2}$

   Write the following as column vectors

   a) **a** – **b**

   b) 4**a** + 2**b**

3. Three vectors are listed below with some missing values

$$\boldsymbol{a} = \binom{3}{2} \quad \boldsymbol{b} = \binom{d}{e} \quad \boldsymbol{c} = \binom{1}{f}$$

   Use the following equations to find the value of d, e and f:

$$\boldsymbol{a} + \boldsymbol{b} = \binom{3}{0}$$

$$2\boldsymbol{c} + \boldsymbol{b} = \binom{2}{2}$$

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.